Lie–algebra expansions, Chern–Simons theories and the Einstein–Hilbert lagrangian

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ABSTRACT

Starting from gravity as a Chern–Simons action for the AdS algebra in five dimensions, it is possible to deform the theory through an expansion of the Lie algebra that leads to a system consisting of the Einstein–Hilbert action plus nonminimally coupled matter. The deformed system is gauge invariant under the Poincaré group enlarged by an Abelian ideal. Although the resulting action naively looks like General Relativity plus corrections due to matter sources, it is shown that the nonminimal couplings produce a radical departure from GR. Indeed, the dynamics is not continuously connected to the one obtained from Einstein–Hilbert action. In a matter–free configuration and in the torsionless sector, the field equations are too strong a restriction on the geometry as the metric must satisfy both the Einstein and pure Gauss–Bonnet equations. In particular, the five-dimensional Schwarzschild geometry fails to be a solution; however, configurations corresponding to a brane-world with positive cosmological constant on the worldsheet are admissible when one of the matter fields is switched on. These results can be extended to higher odd dimensions.

arXiv:hep-th/0605174 v1 17 May 2006

CECS-PHY-06/09
hep-th/0605174 May 2006

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1 Introduction

Field theories in higher dimensions are nowadays a natural part of the high energy physics toolkit. Having accepted the possibility of higher dimensions makes it worthwhile to consider theories with structures that are inherently higher dimensional, and not just a lifting of those that exist in four dimensions. In particular, gravitation theories which are not just the dimensional continuation of the Einstein–Hilbert action deserve special attention. This is the case of the Gauss–Bonnet action, which only exists for $d \geq 5$ dimensions, and shares the basic assumptions of General Relativity, namely, general covariance and second order field equations for the metric. In general, for $d > 4$, the generic theory that accomplishes these criteria has higher powers of curvature, and is given by the Lovelock action [1].

As a purely metric theory, the Lovelock action describes the same number of degrees of freedom as general relativity (GR) [2]. However, in the first order formalism, where the vielbein and the spin connection are independently varied, there could be extra degrees of freedom associated to the fact that the torsion may not vanish in vacuum [3], unlike in GR. This makes it natural to introduce extra terms containing torsion in the lagrangian [4]. It is also natural to wonder whether these theories admit a locally supersymmetric extension. This was shown to be possible for a very particular case of Einstein–Gauss–Bonnet gravity in five dimensions [5], where the coupling of the Gauss–Bonnet term is chosen so that the lagrangian can be viewed as a Chern–Simons (CS) form for the super AdS_5 group. It has also been recently shown that Euclidean gravity as a CS theory in five dimensions can be seen to emerge from a CS theory in $0 + 1$ dimensions in the large N limit [6].

The supersymmetric extension of the lagrangian defined purely by the term with the highest power of curvature in any odd dimension, can also be seen as a CS form for the super Poincaré algebra with a "central extension" [7, 8, 9].

The supersymmetric extensions of the Lovelock theories for higher odd dimensions with a negative cosmological constant can be achieved provided the gravitational sector is supplemented with appropriate torsion terms so that the entire action can be viewed as a CS form for the minimal super AdS group [10, 11].

It is worth highlighting that in eleven-dimensions, apart from the standard Cremmer–Julia–Scherk supergravity theory [12], there exists an AdS supergravity whose gravitational sector has higher powers in the curvature. As pointed out in Ref. [10] this theory has $OSp(1|32)$ gauge invariance, and some sectors of it might be related to the low energy limit of M–theory if one identifies the totally antisymmetric part of the contorsion with the 3-form. Its dual 6-form is also present, and hence, the theory not only has the potential to contain standard supergravity, but also some kind of dual version of it.

This suggestion has been further developed in [13], where it is proposed that M–theory could be related to a Chern–Simons theory with an enlarged gauge group, namely, $OSp(1|32)\times OSp(1|32)$. However, to this date, the contact with the Cremmer–Julia–Scherk theory remains elusive (see also [14, 15]).

On the other hand, a general method to expand a Lie (super) algebra allows to consistently deform a Chern–Simons theory with a given Lie algebra into another one whose Lie algebra has more generators [16]. It is simple to see that applying this method to the eleven-dimensional AdS supergravity yields a deformed theory whose gravitational sector contains the Einstein–Hilbert action nonminimally coupled to a host of additional matter fields. This proposal has been explicitly carried out in [17], where it was found that the field equations of the deformed theory in vacuum do not reduce to the Einstein field equations, but nevertheless it was shown that the M-waves of standard eleven-dimensional supergravity are also solutions of the new theory.

The purpose of this note is to capture the source of the clash between the deformed theory and GR in a simplified setting. We consider gravity as a CS action for the AdS algebra in five dimensions, which is the simplest non-trivial case where the problem arises. Deforming the theory through an expansion of the Lie algebra leads to a CS system that is gauge invariant under the Poincaré group with an Abelian ideal. The resulting action consists of the Einstein–Hilbert term plus other terms containing matter nonminimally coupled to the curvature. These extra terms do not correspond to corrections of GR since although the action reduces to Einstein–Hilbert when the matter fields are switched off, the field equations don't. This is a generic feature of nonminimal couplings, which produce strong deviations from GR. Indeed, in the torsionless sector in vacuum, the geometry must satisfy both the Einstein and pure Gauss–Bonnet equations simultaneously. These restrictions are so strong as to rule out, for instance, the five-dimensional Schwarzschild solution, but not a *pp*-wave.

When matter fields are switched on, however, configurations corresponding to a brane-world with positive cosmological constant on the worldsheet are admissible. Curiously, the metric for this class of solutions also solves the field equations of the supergravity theories with local Poincaré invariance in vacuum, as discussed in [9].

In dimensions $d > 5$, the application of the expansion method as in [16] gives rise to an extension of the Poincaré algebra by a large number of extra generators, which no longer form an Abelian ideal. As a consequence, in the torsionless vacuum sector, in addition to the Einstein and pure Gauss–Bonnet equations, all the possible field equations coming from each term in the Lovelock action must be satisfied. This means that, as the dimensions increase, the clash with General Relativity becomes unsurmountable. However, in all cases *pp*-waves are always solutions of the deformed systems.

2 Five-dimensional Chern–Simons AdS gravity and its expansion

Nonabelian Chern-Simons theories in five dimensions are given by a Lagrangian L such that

$$dL = \langle \mathbf{F} \wedge \mathbf{F} \wedge \mathbf{F} \rangle \ , \tag{1}$$

where $\mathbf{F} = \mathbf{dA} + \mathbf{A} \wedge \mathbf{A}$ is the field strength (curvature) and $\mathbf{A}$ is the nonabelian gauge field. Here the bracket $\langle \cdots \rangle$ stands for a third rank invariant tensor of the Lie algebra. For the five-dimensional AdS algebra the connection reads

$$\mathbf{A} = e^a \, J_a + \frac{1}{2} \, \omega^{ab} \, J_{ab} \ , \tag{2}$$

where J_{ab} and J_a are the generators of Lorentz transformations and AdS boosts, respectively, e^a is the vielbein and ω^{ab} is the spin connection. The only nonvanishing components of the bracket are

$$\langle J_{ab}, J_{cd}, J_e \rangle = \frac{4}{3} \epsilon_{abcde} \ . \tag{3}$$

Thus, the action is the Einstein–Gauss–Bonnet theory with fixed relative couplings, up to surface terms,

$$I = \kappa \int \epsilon_{abcdf} \left(R^{ab} \wedge R^{cd} \wedge e^f + \frac{2}{3} R^{ab} \wedge e^c \wedge e^d \wedge e^f + \frac{1}{5} e^a \wedge e^b \wedge e^c \wedge e^d \wedge e^f \right) \ , \tag{4}$$

where the AdS radius has been fixed to one, and $R^{ab} = d\omega^{ab} + \omega^a_{\ c} \wedge \omega^{cb}$ is the curvature two-form.

Applying the expansion method of [16] up to second order, the connection is deformed as

$$\mathbf{A} \to \widetilde{\mathbf{A}} = e^a \, P_a + \frac{1}{2} \, \omega^{ab} \, J_{ab} + h^a \, Z_a + \frac{1}{2} \, \kappa^{ab} \, Z_{ab} \ . \tag{5}$$

This connection is the gauge field for an extension of the Poincaré algebra by an Abelian ideal spanned by $\{Z_a, Z_{ab}\}$, so that the commutation relations of the full algebra are given by

$$\begin{array}{l} [P_a, P_b] = Z_{ab} \ , \qquad [J_{ab}, P_c] = P_a \, \eta_{bc} - P_b \, \eta_{ac} \ , \\ [J_{ab}, J_{cd}] = -J_{ac} \, \eta_{bd} + J_{bc} \, \eta_{ad} - J_{bd} \, \eta_{ac} + J_{ad} \, \eta_{bc} \ , \\ [Z_a, Z_b] = [Z_{ab}, Z_c] = [Z_{ab}, Z_{cd}] = [P_a, Z_b] = 0 \ , \\ [Z_{ab}, P_c] = [J_{ab}, Z_c] = Z_a \, \eta_{bc} - Z_b \, \eta_{ac} \ , \\ [J_{ab}, Z_{cd}] = -Z_{ac} \, \eta_{bd} + Z_{bc} \, \eta_{ad} - Z_{bd} \, \eta_{ac} + Z_{ad} \, \eta_{bc} \ . \end{array} \tag{6}$$

The curvature two-form corresponding to the connection (5), is obtained as

$$\begin{aligned} \mathbf{F} \to \widetilde{\mathbf{F}} &= T^a \, P_a + \frac{1}{2} \, R^{ab} \, J_{ab} + \left[Dh^a + \kappa_b^{\ a} \wedge e^b \right] \, Z_a + \frac{1}{2} \left[D\kappa^{ab} + e^a \wedge e^b \right] Z_{ab} \ , \\ &= T^a \, P_a + \frac{1}{2} \, R^{ab} \, J_{ab} + \tilde{F}^a \, Z_a + \frac{1}{2} \, \tilde{F}^{ab} \, Z_{ab} \ , \end{aligned} \tag{7}$$

where D is the covariant derivative with respect to the Lorentz piece of the connection, and $T^a = De^a$. The expansion method ensures that one can consistently isolate the second order term in the expansion of the Lagragian, which corresponds to a CS theory for the deformed algebra, *i. e.*,

$$d\widetilde{L} = \left\langle \widetilde{\mathbf{F}} \wedge \widetilde{\mathbf{F}} \wedge \widetilde{\mathbf{F}} \right\rangle \ , \tag{8}$$

where now the bracket is defined as

$$\langle J_{ab}, J_{cd}, Z_f \rangle = \langle J_{ab}, Z_{cd}, P_f \rangle = \frac{4}{3} \, \epsilon_{abcdf} \ . \tag{9}$$

The resulting deformed action is locally invariant under gauge transformations generated by the algebra (6). If expressed in terms of the components of the gauge field the action reads (we put $\kappa = 1$),

$$\widetilde{I} = \int \epsilon_{abcdf} \left(\frac{2}{3} R^{ab} \wedge e^c \wedge e^d \wedge e^f + R^{ab} \wedge R^{cd} \wedge h^f + 2 R^{ab} \wedge \kappa^{cd} \wedge T^f \right) \ , \tag{10}$$

up to surface terms.

A shortcut to obtain the deformed action from (4) is as follows: shift the fields as

$$\omega^{ab} \rightarrow \omega^{ab} + \ell^{-2}\, \kappa^{ab} \ , \tag{11}$$

$$e^{a} \rightarrow \ell^{-1}\, e^{a} + \ell^{-3}\, h^{a} \ . \tag{12}$$

The deformed action (10) is then obtained replacing the shifted fields in the original action (4) and retaining the terms of order ℓ^{-3}. As shown in the Appendix, it is straightforward to generalize this shortcut to apply the expansion procedure in higher dimensions.

It is apparent from the action (10) that if one identifies the field e^a with the vielbein, the system consists of the Einstein–Hilbert action plus nonminimally coupled matter fields given by h^a and κ^{ab}. Although the resulting action naively looks like general relativity plus corrections due to the matter sources, one can see that the nonminimal couplings are so intricate as to produce a radical departure from general relativity. These extra terms do not correspond to corrections of General Relativity since although the action reduces to Einstein–Hilbert when the matter fields are switched off, the field equations don't. Indeed, varying the action (4) with respect to the vielbein, the spin connection and the additional bosonic fields, h^a and κ^{ab}, the field equations read respectively,

$$\epsilon_{abcdf}\, R^{ab} \wedge e^{c} \wedge e^{d} = -\epsilon_{abcdf}\, R^{ab} \wedge D\kappa^{cd} \ , \tag{13}$$

$$\epsilon_{abcdf} \left(\widetilde{F}^{cd} \wedge T^{f} + R^{cd} \wedge \widetilde{F}^{f} \right) = 0 \ , \tag{14}$$

$$\epsilon_{abcdf}\, R^{ab} \wedge R^{cd} = 0 \ , \tag{15}$$

$$\epsilon_{abcdf}\, R^{ab} \wedge T^{c} = 0 \ . \tag{16}$$

In a matter–free configuration and in the torsionless sector, the metric must satisfy simultaneously the Einstein and the pure Gauss–Bonnet equations,

$$\epsilon_{abcdf}\, R^{ab} \wedge e^{c} \wedge e^{d} = 0 \ , \tag{17}$$

$$\epsilon_{abcdf}\, R^{ab} \wedge R^{cd} = 0 \ , \tag{18}$$

which is certainly a severe restriction on the geometry, and not just a correction to General Relativity. In particular, it is simple to see that the only spherically symmetric solution satisfying both equations is flat spacetime (see *e. g.*, [18]), and hence the five-dimensional Schwarzschild geometry is ruled out as a solution.

It is amusing to see that requiring the metric to solve equations (17) and (18) simultaneously, is too strong so as to rule out a spherically symmetric black hole, but not a *pp*-wave. Indeed, it is simple to see that *pp*-wave solutions of the Einstein equations solve independently (18) as well as any equation constructed from powers of the Riemann tensor with two free indices. This is because the Riemann tensor for a *pp*-wave is orthogonal to a (covariantly constant) null vector on all its indices, so that any term involving contractions of the Riemann tensor, being quadratic or of higher degree identically vanishes. This is the underlying reason of why *pp*-waves solve "stringy corrections" to General Relativity which involve higher powers of the curvature to all orders (see *e. g.*, [19]).

Even though the field equations turn out to be too restrictive for the metric, they allow certain freedom when matter fields are switched on. The purpose of the next subsection is to show that there exist configurations corresponding to a brane-world with positive cosmological constant on the worldsheet, supported by matter fields which are discontinuous, but not singular, across the brane.

2.1 Four–dimensional brane world solution

Switching on the bosonic field κ^{ab}, the field equations in the torsionless sector read

$$\epsilon_{abcdf}\, R^{ab} \wedge e^c \wedge e^d = -\epsilon_{abcdf}\, R^{ab} \wedge D\kappa^{cd} \ , \tag{19}$$

$$\epsilon_{[a|cdfg}\, R^{cd} \wedge \kappa^{fg} \wedge e_{|b]} = 0 \ , \tag{20}$$

$$\epsilon_{abcdf}\, R^{ab} \wedge R^{cd} = 0 \ . \tag{21}$$

Let us consider a domain wall of the form

$$ds^2 = e^{2f(|z|)}\ \left(dz^2 + \tilde{g}^{(4)}_{\mu\nu} dx^\mu dx^\nu\right) \ ,$$

where $\tilde{g}^{(4)}_{\mu\nu} = \tilde{g}^{(4)}_{\mu\nu}(x)$ is the metric on the worldsheet. The vielbein can be chosen as

$$e^m = e^{f(|z|)}\, \tilde{e}^m \ , \qquad e^4 = e^{f(|z|)}\, dz \ , \tag{22}$$

where $\tilde{e}^m = \tilde{e}^m(x)$, with $m = 0, \cdots, 3$, is the vielbein along the worldsheet. The only non vanishing component of the bosonic field κ^{ab} is assumed to be of the form

$$\kappa^{m4} = g(z)\, \tilde{e}^m \ . \tag{23}$$

It is easy to see that the field equations (19-21) are solved provided

$$\tilde{R}^{mn} = (f'(|z|))^2\, \tilde{e}^m \wedge \tilde{e}^n \ , \qquad e^{-2f(|z|)} f'(|z|) = \frac{1}{2g(z)} \ , \tag{24}$$

where $\tilde{R}^{mn}$ is the curvature two-form along the worldsheet. This means that $(f'(|z|))^2$ must be a positive constant, which in turn implies that $f(|z|) = -\xi|z|$. Thus, the geometry of the domain wall acquires the form

$$ds^2 = e^{-2\xi|z|}\ \left(dz^2 + \tilde{g}^{(4)}_{\mu\nu} dx^\mu dx^\nu\right) \ , \tag{25}$$

where the worldsheet metric of the four-dimensional brane-world must be locally a de Sitter spacetime with radius $\ell = \xi^{-1}$, and moreover, the non-vanishing components of the bosonic field read

$$\kappa^{m4} = -\frac{1}{2\xi} \mathrm{sgn}(z)\, e^{-2\xi|z|}\, \tilde{e}^m \ . \tag{26}$$

3 Summary and discussion

In this note, it is shown that deforming the Chern-Simons theory for AdS gravity according to the expansion procedure of [16] is not sufficient to produce a direct link with standard General Relativity. The fact that the Einstein-Hilbert term appears in the action is just a mirage, since actually the nonminimally coupled matter fields cannot be regarded as small corrections. In fact, the dynamics suffers a radical departure from General Relativity so that it is not continuously connected to the one obtained from the Einstein–Hilbert action. Indeed, for vacuum configurations without torsion, the field equations give severe restrictions on the geometry, so that in particular, the metric must simultaneously satisfy the Einstein as well as the pure Lovelock field equations. This precludes the existence of spherically symmetric black holes, but does not rule out *pp*-waves.

In the five-dimensional case, it is shown that configurations corresponding to a brane-world with positive cosmological constant on the worldsheet are admissible when one of the matter fields is switched on. Curiously, the metric for this class of solutions also solves the field equations of the supergravity theories with local Poincaré invariance in vacuum[1], as discussed in [9].

It goes without saying that the procedure can also be applied for the locally supersymmetric extensions of these theories; however, it is clear that supersymmetry would not improve the situation.

One can notice that there is an arbitrary element in the identification of the gauge fields with spacetime geometry. For instance, in the five dimensional case the connection (5) has two possible candidates to be identified with the vielbein, namely, the fields e^a and h^a, since both transform as vectors under local Lorentz transformations. Choosing e^a, makes the Einstein-Hilbert term to appear in the action, but as discussed here, this choice brings in, apart from the Einstein equations (17), the Gauss-Bonnet ones (18) to be simultaneously fulfilled by the geometry in vacuum. Alternatively, if one identifies h^a as the vielbein, the Einstein-Hilbert term does not appear in the action, but the theory in vacuum is well defined in the sense that the geometry must satisfy only the Gauss-Bonnet equations, and the metric is not overdetermined as for the other choice. The asymmetry between these two choices comes from the fact that in the first case, the corresponding generators P_a are actually pseudo translations since $[P_a, P_b] = Z_{ab}$, which brings in the extra field equations to be required in vacuum. For the other choice, instead, the corresponding generators Z_a commute among themselves and can be identified as the translation part of the Poincaré group; so, when matter fields are switched off, there are no extra field equations for the geometry. Thus, for this choice the nonminimally coupled matter fields actually correspond to corrections that can be consistently switched off, but for the Gauss–Bonnet theory.

From the last remark one can observe that in general, the expansion method generates many fields that are candidates to be identified with the vielbein, and for each choice a completely different gravitational sector arises. An arbitrary choice is going to suffer of the same overdeterminacy of the spacetime metric as it occurs for the choice that makes the Einstein-Hilbert term to appear in the action. However, there is always one special choice that is free from this problem. For this choice, the gravitational sector is described by a lagrangian that is the dimensional continuation of the Euler density of one dimension below. This is just the CS form for the Poincaré group.

The expansion methods described here are also useful to connect an eleven-dimensional

[1]Note that this class of solutions possesses a jump in the extrinsic curvature without the need of a thin shell as matter source. This can be seen to be allowed by the generalization of the Israel junction conditions for this kind of theories [20].

AdS supergravity theory with a gauge theory for the M-algebra [21].

Acknowledgments

JDE is pleased to thank J. M. Izquierdo for interesting discussions and comments. This work has been partially supported by grants 1051084, 1060831, 1040921, 1051056, 1061291 from FONDECYT. The work of JDE has been supported in part by ANPCyT under grant PICT 2002 03-11624, by the FCT grant POCTI/FNU/38004/2001, by MCyT, FEDER and Xunta de Galicia under grant FPA2005-00188, and by the EC Commission under grants HPRN-CT-2002-00325 and MRTN-CT-2004-005104, and by grant CONICYT/SECYT CH/PA03-EIII/014. Institutional support to the Centro de Estudios Científicos (CECS) from Empresas CMPC is gratefully acknowledged. CECS is funded in part by grants from the Millennium Science Initiative, Fundación Andes and the Tinker Foundation.

Appendix. Extension to arbitrary odd dimensions

As explained in the five-dimensional case, the expansion method of [16] can be implemented for gravity as a Chern-Simons theory for the AdS group in higher odd dimensions, through a similar shortcut consisting on shifting the fields as in Eqs. (11), (12). The deformed action is then obtained by replacing the shifted fields in the original action. For dimensions $d = 2n + 1$, the generalized shift reads

$$\omega^{ab} \to \omega^{ab} + \sum_{k=1}^{n-1} \ell^{-2k}\, \kappa^{ab}_{(k)}\ , \qquad e^a \to \ell^{-1}\, e^a + \sum_{k=1}^{n-1} \ell^{-(2k+1)}\, h^a_{(k)}\ , \tag{27}$$

where $h^a_{(k)}$, and $\kappa^{ab}_{(k)}$ are extra bosonic one-form fields. The gravitational action, instead, is given by

$$I_{AdS} = \int \sum_{p=0}^{n} \frac{1}{(2n+1-2p)} \binom{n}{p}\, \mathcal{L}^{(p)} \tag{28}$$

where $\mathcal{L}^{(p)}$ are the dimensional continuation of the Euler forms

$$\mathcal{L}^{(p)} = \epsilon_{a_1\cdots a_{2n+1}} R^{a_1a_2} \wedge \cdots \wedge R^{a_{2p-1}a_{2p}} \wedge e^{a_{2p+1}} \wedge \cdots \wedge e^{a_{2n+1}}\ . \tag{29}$$

The deformed action is obtained after substituting (27) in (28) and retaining the terms of order ℓ^{2-d}. Its explicit form is not particularly illuminating, but, as expected, contains the Einstein–Hilbert term

$$\begin{aligned} I^{2n+1} \ &= \ \int \sum_{p=2}^{n} \frac{\binom{n}{p}}{(2n+1-2p)} \epsilon_{a_1\cdots a_{2p}a_{2p+1}\cdots a_{2n+1}} \sum_{k=0}^{p}\sum_{k_1=0}^{k}\cdots\sum_{k_{(n-2)}=0}^{k_{(n-3)}} \mathcal{C}_{p,k,k_1,\cdots,k_{(n-2)}} \\ & R^{k_{(n-2)}} \left[\prod_{j=1}^{n-2} R_{(j)}^{k_{(n-2-j)}-k_{(n-1-j)}}\right] R^{p-k}_{(n-1)} \sum_{r=0}^{2n+1-2p}\sum_{r_1=0}^{r}\cdots\sum_{r_{(n-2)}=0}^{r_{(n-3)}} \mathcal{C}_{2n+1-2p,r,r_1,\cdots,r_{(n-2)}} \\ & e^{r_{(n-2)}} \left[\prod_{i=1}^{n-2} h_{(i)}^{r_{(n-2-i)}-r_{(n-1-i)}}\right] h^{2n+1-2p-r}_{(n-1)} + \frac{n}{2n-1} \epsilon_{a_1\cdots a_{2n+1}} R^{a_1a_2} e^{a_3}\cdots e^{a_{2n+1}}\ . \end{aligned} \tag{30}$$

Here $R^q_{(j)}$ means the wedge product of q curvatures defined by

$$R^{ab}_{(k)} = \sum_{i,j=1(i+j=k)}^{(k-1)} D(\omega)\,\kappa^{ab}_{(k)} + \kappa^{a}_{(i)c} \wedge \kappa^{cb}_{(j)} \ .$$

with the appropriate Lorentz indices, *i.e.* $R^{a_{s+1}a_{s+2}}_{(j)} \cdots R^{a_{2q-1-s}a_{2q-s}}_{j}$ and the same convention is adopted for $h^q_{(j)}$ that is the wedge product of q one-form fields $h_{(j)}$. [2] Besides, we omit the wedge product symbol.

However, this action defines a Chern-Simons theory for the connection

$$\mathcal{A} = e^a\,P_a + \frac{1}{2}\,\omega^{ab}\,J_{ab} + \sum_{k=1}^{n-1} h^a_{(k)}\,Z^{(k)}_a + \frac{1}{2}\sum_{k=1}^{n-1} \kappa^{ab}_{(k)}\,Z^{(k)}_{ab} \ ,$$

where one needs to include the aditional generators $Z^{(i)}_a$, and $Z^{(i)}_{ab}$ with $i = 1, \cdots, n-1$. The resulting algebra has the following commutation relations

$$\begin{aligned}
&[P_a, P_b] = Z^{(1)}_{ab} \ , \qquad [J_{ab}, P_c] = P_a\,\eta_{bc} - P_b\,\eta_{ac} \ , \qquad [J_{ab}, J_{cd}] = -J_{ac}\,\eta_{bd} + \cdots \ , \\
&\left[Z^{(i)}_a, Z^{(j)}_b\right] = Z^{(i+j+1)}_{ab} \ , \qquad \left[Z^{(i)}_{ab}, Z^{(j)}_c\right] = Z^{(i+j)}_a\,\eta_{bc} - Z^{(i+j)}_b\,\eta_{ac} \ , \\
&\left[Z^{(i)}_{ab}, Z^{(j)}_{cd}\right] = -Z^{(i+j)}_{ac} + \cdots \ , \qquad \left[P_a, Z^{(i)}_b\right] = Z^{(i+1)}_{ab} \ , \\
&\left[Z^{(i)}_{ab}, P_c\right] = Z^{(i)}_a\,\eta_{bc} - Z^{(i)}_b\,\eta_{ac} = \left[J_{ab}, Z^{(i)}_c\right] \ , \qquad \left[J_{ab}, Z^{(i)}_{cd}\right] = -Z^{(i)}_{ac}\eta_{bd} + \cdots
\end{aligned} \tag{31}$$

and the nonvanishing components of the $(n+1)$th–rank invariant tensor are given by

$$\begin{aligned}
&\left\langle J_{a_1a_2}, \cdots, J_{a_{2n-1}a_{2n}}, Z^{(n-1)}_{a_{2n+1}} \right\rangle = \frac{2^n}{n+1}\,\epsilon_{a_1\cdots a_{2n+1}} \ , \\
&\left\langle J_{a_1a_2}, \cdots, J_{a_{2i-1}a_{2i}}, Z^{(j_1)}_{a_{2i+1}a_{2i+2}}, \cdots, Z^{(j_q)}_{a_{2n-1}a_{2n}}, Z^{(p)}_{a_{2n+1}} \right\rangle = \frac{2^n}{n+1}\,\epsilon_{a_1\cdots a_{2n+1}} \ , \\
&\left\langle J_{a_1a_2}, \cdots, J_{a_{2k-1}a_{2k}}, Z^{(l_1)}_{a_{2k+1}a_{2k+2}}, \cdots, Z^{(l_r)}_{a_{2n-1}a_{2n}}, P_{a_{2n+1}} \right\rangle = \frac{2^n}{n+1}\,\epsilon_{a_1\cdots a_{2n+1}} \ .
\end{aligned}$$

Here the numbers of generators i and q (resp. k and r) satisfy $i+q=n$ (resp. $k+r=n$) and the other parameters are chosen such that $2(j_1+\cdots j_q)+2(p-n+1)=0$, (resp. $2(l_1+\cdots l_r)+2(1-n)=0$).

In the torsionless sector of the theory and in absence of matter fields, the field equations

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[2] We have also introduced the quantity $\mathcal{C}_{r,s,s_1,\cdots,s_N} \equiv \binom{r}{s}\binom{s}{s_1}\cdots\binom{s_{N-1}}{s_N}$. The parameters are restricted as follows: for each fixed p such that $2 \le p \le n$,

$$j\left(k_{(n-2-j)} - k_{(n-1-j)}\right) + i\left(r_{(n-2-i)} - r_{(n-1-j)}\right) + (n-1)(2n-p-k-r+1) = p-1 \ ,$$

where the integers i and j run over $1, \cdots, (n-2)$ with $k_{(n-2)} \le k_{(n-3)} \le \cdots \le k_1 \le k \le p$ and $r_{(n-2)} \le r_{(n-3)} \le \cdots \le r_1 \le r \le (2n+1-2p)$.

are given by

$$
\begin{aligned}
&\epsilon_{a_1\cdots a_{2n+1}} R^{a_1a_2}\wedge\cdots\wedge R^{a_{2n-1}a_{2n}}=0\ ,\\
&\epsilon_{a_1\cdots a_{2n+1}} R^{a_1a_2}\wedge\cdots\wedge R^{a_{2n-3}a_{2n-2}}\wedge e^{a_{2n-1}}\wedge e^{a_{2n}}=0\ ,\\
&\vdots\\
&\epsilon_{a_1\cdots a_{2n+1}} R^{a_1a_2}\wedge e^{a_3}\wedge\cdots\wedge e^{a_{2n}}=0\ ,
\end{aligned}
$$

which means that the metric must satisfy simultaneously the Einstein as well as all possible "pure Lovelock" equations. This is an even more severe restriction on the geometry than in the five-dimensional case, and not just a correction to General Relativity. Analogously, it is simple to see that the only spherically symmetric solution satisfying all the equations is flat spacetime (see *e.g.*, [18]), so that Schwarzschild solution is ruled out. However, it is simple to see that *pp*-wave solutions of the Einstein equations solve the whole system.

References

[1] D. Lovelock, "The Einstein Tensor And Its Generalizations," J. Math. Phys. **12**, 498 (1971).

[2] C. Teitelboim and J. Zanelli, "Dimensionally continued topological gravitation theory in Hamiltonian form," Class. Quant. Grav. **4**, L125 (1987).

[3] R. Troncoso and J. Zanelli,"Higher dimensional gravity and local AdS symmetry," Class. Quant. Grav. **17**, 4451 (2000) [arXiv:hep-th/9907109].

[4] A. Mardones and J. Zanelli,"Lovelock-Cartan Theory Of Gravity," Class. Quant. Grav. **8**, 1545 (1991).

[5] A. H. Chamseddine, "Topological gravity and supergravity in various dimensions", Nucl. Phys. **B346**, 213 (1990).

[6] V. P. Nair, "The Chern-Simons one-form and gravity on a fuzzy space," arXiv:hep-th/0605008; " The matrix Chern-Simons one-form as a universal Chern-Simons theory," arXiv:hep-th/0605007.

[7] M. Banados, R. Troncoso and J. Zanelli, "Higher dimensional Chern-Simons supergravity," Phys. Rev. **D54**, 2605 (1996) [arXiv:gr-qc/9601003].

[8] M. Hassaïne, R. Olea and R. Troncoso, "New N = 2 supergravity with local Poincare invariance in nine dimensions," Phys. Lett. **B599**, 111 (2004) [arXiv:hep-th/0210116].

[9] M. Hassaïne, R. Troncoso and J. Zanelli, "Poincaré invariant gravity with local supersymmetry as a gauge theory for the M-algebra," Phys. Lett. **B596**, 132 (2004) [arXiv:hep-th/0306258]; "11D Supergravity as a gauge theory for the M-algebra," PoS **WC2004**, 006 (2005) [arXiv:hep-th/0503220].

[10] R. Troncoso and J. Zanelli, “New gauge supergravity in seven and eleven dimensions,” Phys. Rev. **D58**, 101703 (1998) [arXiv:hep-th/9710180]. “Gauge supergravities for all odd dimensions,” Int. J. Theor. Phys. **38**, 1181 (1999) [arXiv:hep-th/9807029].

[11] R. Troncoso and J. Zanelli, “Chern-Simons supergravities with off-shell local superalgebras,” [arXiv:hep-th/9902003]. J. Zanelli, “Lecture notes on Chern-Simons (super-) gravities,” [arXiv:hep-th/0502193].

[12] E. Cremmer, B. Julia and J. Scherk, “Supergravity Theory In 11 Dimensions,” Phys. Lett. **B76**, 409 (1978).

[13] P. Hořava, “M-theory as a holographic field theory,” Phys. Rev. **D59**, 046004 (1999) [arXiv:hep-th/9712130].

[14] M. Bañados, “The linear spectrum of OSp(32|1) Chern-Simons supergravity in eleven dimensions,” Phys. Rev. Lett. **88**, 031301 (2002) [arXiv:hep-th/0107214].

[15] H. Nastase, “Towards a Chern-Simons M theory of OSp(1|32) x OSp(1|32),” arXiv:hep-th/0306269.

[16] J. A. de Azcárraga, J. M. Izquierdo, M. Picón and O. Varela, “Generating Lie and gauge free differential (super)algebras by expanding Maurer-Cartan forms and Chern-Simons supergravity,” Nucl. Phys. **B662**, 185 (2003). [arXiv:hep-th/0212347].

[17] J. M. Izquierdo, “Expansions of Lie superalgebras and D=11 Chern–Simons supergravity,” in *Symmetries in gravity and field theory: Homenaje al profesor José Adolfo de Azcárraga*, V. Aldaya, J. Cervero and P. García (eds), Ediciones Universidad de Salamanca, Salamanca, 2004.

[18] G. Giribet, J. Oliva and R. Troncoso,“Simple compactifications and black p-branes in Gauss-Bonnet and Lovelock theories,” JHEP**0605**:007(2006). [arXiv:hep-th/0603177].

[19] G. T. Horowitz and A. R. Steif, “Space-Time Singularities In String Theory,” Phys. Rev. Lett. **64**, 260 (1990).

[20] E. Gravanis and S. Willison, “Intersecting hyper-surfaces in dimensionally continued topological density gravitation,” J. Math. Phys. **45**, 4223 (2004) [arXiv:hep-th/0306220]; “Israel conditions for the Gauss-Bonnet theory and the Friedmann equation on the brane universe,” Phys. Lett. **B562**, 118 (2003) [arXiv:hep-th/0209076].

[21] M. Hassaïne, R. Troncoso and J. Zanelli, “Emergence of the M-algebra from eleven-dimensional AdS supergravity”, Preprint CECS-PHY-06/08 (to appear).